Hyperbolic tetrad formulation of the Einstein equations for numerical relativity

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The tetrad-based equations for vacuum gravity published by Estabrook, Robinson, and Wahlquist are simplified and adapted for numerical relativity. We show that the evolution equations as partial differential equations for the Ricci rotation coefficients constitute a rather simple first-order symmetrizable hyperbolic system, not only for the Nester gauge condition on the acceleration and angular velocity of the tetrad frames considered by Estabrook *et al.*, but also for the Lorentz gauge condition of van Putten and Eardley and for a fixed gauge condition. We introduce a lapse function and a shift vector to allow general coordinate evolution relative to the timelike congruence defined by the tetrad vector field.

I. INTRODUCTION

Gravitational wave detection is soon to become a reality. However, the scientific community has not yet been able to calculate gravitational waveforms from the most likely source, namely the violent and dynamic merger phase of binary black hole collisions [1]. We hope to contribute to this effort with detailed small-scale studies. Our first such study of 1D colliding gravitational plane waves [2] indicated that two important factors which improve the accuracy and stability of the numerical calculations are the hyperbolicity of the equations and evolving variables which are related to physical quantities. As a way to generalize our results to black hole spacetimes and higher dimensions, we are investigating a tetrad approach based on the formalism published by Estabrook, Robinson, and Wahlquist (ERW) [3]. In this paper, we present a modified version of the formalism, adapted for numerical relativity. Subsequent papers will present our numerical results.

The standard approach to vacuum numerical relativity is a 3+1 decomposition, of the type introduced by Arnowitt, Deser, and Misner [4]. The 3+1 formulations slice four-dimensional spacetime into three-dimensional spacelike hypersurfaces (see [5] for a recent review). They evolve the spatial metric and extrinsic curvature of the hypersurfaces, which are expressed in a coordinate basis. The evolution of the coordinates is described by a lapse function and shift vector, which may or may not be dynamic.

A tetrad formulation uses orthonormal basis vectors, e_α ($\alpha = 0, 1, 2, 3$), which describe local Lorentz frames. The spacetime metric, the dot product of the basis vectors, is everywhere the Minkowski metric, $g_{\alpha\beta} = e_\alpha \cdot e_\beta = \eta_{\alpha\beta}$. The timelike vector field of the orthonormal frames, e_0, defines a preferred timelike congruence, to which it is tangent. The spatial triad vectors in a particular rotational orientation with respect to e_0 are e_1, e_2, and e_3. The dual basis of orthonormal one-forms is e^α such that $\langle e^\alpha, e_\beta \rangle = \delta^\alpha_\beta$ and $e^\alpha \cdot e^\beta = \eta^{\alpha\beta}$. As Estabrook and Wahlquist point out [6], these tetrad frames are natural for measuring observable physical quantities.

Most of the recent tetrad formalisms assume that the tetrads are tied to the physically defined flow of a fluid, primarily in the context of cosmology or of interior metrics of rotating stars. Here, we advocate that a tetrad formalism may be useful even in vacuum black hole spacetimes. The variables in a tetrad formalism are the connection coefficients (often called the Ricci rotation coefficients), the tetrad vector components, and, typically, the tetrad components of the Weyl tensor or the Riemann tensor. The spatial coordinates are often assumed from the start to be comoving with the timelike congruence generated by the tetrad field. In dealing with black hole event horizons, however, it is important to allow general choices of coordinates. We do this by introducing a lapse function and a shift vector defined relative to the congruence world lines (see also van Putten and Eardley [7]).

The ERW formulation is a quasi-FOSH (first order symmetric hyperbolic) system. We use the term "quasi" because the directional derivatives along the spatial tetrad legs contain partial time derivatives (also see [8]). The basic quasi-FOSH structure of the equations involves the Weyl tensor components as variables. The quasi-FOSH system also includes equations derived from the Nester gauge conditions [9]. These equations evolve the acceleration of the congruence world lines, a, and the angular velocity of the spatial tetrad vectors relative to Fermi-Walker transport, ω. The tetrad components a_a and ω_a are gauge quantities since the spacetime orientation of the tetrad is not fixed by the spacetime geometry. Both a and ω are orthogonal to e_0, and therefore have only the three spatial tetrad components. In addition to evolution equations, the Nester gauge conditions provide equations which constrain the spatial dependence of the a_a and the ω_a at any given time.

ERW's formalism can be modified to give a particularly simple quasi-FOSH system, as suggested by Estabrook and Wahlquist (EW) [10], by eliminating the Weyl tensor components as separate variables, and adding the Nester constraint equations to the evolution equations.

In this paper, we discuss the derivation of the EW formalism in a way which is perhaps more accessible to those familiar with standard tensor analysis, as opposed to Cartan

differential form analysis. What distinguishes the EW formalism is that the basic quasi-FOSH structure involves only the connection coefficients. Most of the tetrad formulations in the literature include the Weyl tensor or the Riemann tensor to get a simple quasihyperbolic structure. Unless one adds the constraints to the Einstein equations in the particular way we present in this paper, the quasihyperbolic system involving only the connection coefficients is quite complicated. We do not claim that the EW system is simpler or more elegant than those involving the Weyl or Riemann tensor; however, it has fewer variables and fewer constraint modes, which may be advantageous in 3D codes.

After deriving the EW formalism, we proceed to extend it into a form useful for vacuum numerical relativity. We consider two gauge conditions other than the Nester gauge which also give quasi-FOSH systems of equations. These are the Lorentz gauge used by van Putten and Eardley [7], and a fixed gauge, where a_a and ω_a are fixed functions of time and space. Additionally, we allow for a completely arbitrary relationship between the congruence and the hypersurface. Finally, we analyze the true hyperbolicity of the equations when expressed in terms of partial derivatives. The partial differential system contains evolution equations for eighteen nongauge connection coefficients, six gauge quantities (if a dynamic gauge condition is chosen), three components of a vector describing the velocity of the congruence relative to the hypersurface, and nine components of the spatial tetrad vectors projected into a $t=$const hypersurface. These last twelve variables, together with our lapse function and shift vector (which we do not evolve), completely determine the sixteen tetrad vector components. The flux Jacobian of the partial differential system of equations has a complete set of eigenvectors and real eigenvalues, thus satisfying the requirement for hyperbolicity as per LeVeque [11]. Furthermore, these equations are symmetrizable hyperbolic as defined by Lindblom and Scheel [12].

Tetrad formulations for general relativity other than ERW and EW include those by Friedrich [13], Jantzen, Carini, and Bini [14], van Elst and Uggla [15], Choquet-Bruhat and York [8], and van Putten and Eardley [7]. Except for EW, all of these systems include the Weyl or Riemann tensor components as fundamental variables. Friedrich's paper is a definitive discussion of hyperbolicity for both tetrad and 3+1 representations. Janzten, Carini, and Bini give an extensive historical review of the tetrad and 3+1 approaches. They provide a unified framework for the two approaches in what they call "gravitoelectromagnetism," in which, however, they only consider comoving spatial coordinates. van Elst and Uggla's formalism applies only to nonrotating congruences and orthogonal hypersurface slicings. Also, they do not bring the equations into a quasi-FOSH form. The main emphasis of Choquet-Bruhat and York's paper is a tetrad formulation for fluids, where the congruence is aligned with the fluid flow lines, and the spatial coordinates are comoving with the congruence. The acceleration of the congruence world lines is given by the acceleration of the fluid, and the angular velocity variables (our ω_a's) are fixed functions. Choquet-Bruhat and York do briefly mention the vacuum case, in which their system is quasi-FOSH for arbitrary given

acceleration and angular velocity. van Putten and Eardley use the Lorentz gauge condition to obtain second-order wave equations for the connection coefficients. van Putten gives a first-order form of these equations, involving the Riemann tensor as well as the connection coefficients, for his numerical implementation in 1D Gowdy wave vacuum spacetime [16]. The van Putten–Eardley formulation allows for complete freedom in the choice of spacetime foliation.

An example of using orthonormal frames in spatial hypersurfaces is the Ashtekar formulation (with subsequent modifications) [17–21]. Ashtekar follows a 3+1 split in the sense that his variables are defined relative to a hypersurface. However, his description of the geometry of the hypersurface is in terms of orthonormal triads instead of the metric. He uses complex variables to give a compact formalism, requiring the use of reality constraints to recover real spacetime. As subsequently modified by Yoneda and Shinkai, Ashtekar's formulation becomes a FOSH system of partial differential equations. Shinkai and Yoneda [22,23] have published numerical studies using this formulation in 1D plane wave vacuum spacetime.

II. VARIABLES

Throughout this paper, lower case Greek letters denote spacetime indices (0-3), and lower case Latin letters denote only spatial indices (1-3). The letters in the beginning of the alphabets, (α, β, γ, δ, ϵ) and (a, b, c, d, e, f), denote tetrad indices. Mid-alphabet letters (λ, μ, ν) and (i, j, k, l) denote coordinate indices. Repeated indices are summed in all cases.

For an orthonormal tetrad of basis vector fields, there are twenty-four distinct connection coefficients. These coefficients are scalar fields under coordinate transformations, and are defined as

$$\Gamma_{\alpha\beta\gamma} = e_\alpha \cdot \nabla_\gamma \, e_\beta = -\Gamma_{\beta\alpha\gamma}, \qquad (1)$$

with ∇ the covariant derivative operator. The $\Gamma_{\alpha\beta\gamma}$ are the same as Ricci rotation coefficients (see Wald [24]). They can be written in terms of commutators of the basis vectors:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\{\langle e^\delta, [e_\alpha, e_\beta]\rangle \eta_{\delta\gamma} + \langle e^\delta, [e_\alpha, e_\gamma]\rangle \eta_{\delta\beta}$$
$$- \langle e^\delta, [e_\beta, e_\gamma]\rangle \eta_{\delta\alpha}\}. \qquad (2)$$

Just as in the 3+1 formalisms, it is convenient to make a space-time split in the tetrad formulation. Here, however, the split is relative to the timelike congruence defined by the tetrad rather than the constant-t spacelike hypersurface. The connection coefficients can then be relabeled as 3D quantities with spatial triad indices (see Wahlquist [25] and ERW). To begin,

$$K_{ba} \equiv \Gamma_{a0b}, \qquad (3)$$

where the symmetric part of K_{ba} is the rate of strain of the congruence, and the antisymmetric part,

$$\Omega_a \equiv \frac{1}{2}\varepsilon_{abc}K_{bc}, \qquad (4)$$

is the vorticity of the congruence. If $\Omega_a = 0$, the congruence is hypersurface orthogonal, and K_{ba} is the traditional extrinsic curvature of the orthogonal hypersurface. Note that the sign of K_{ba} here is the same as in ERW and Wald, but opposite to that of Misner, Thorne, and Wheeler [26].

The spatial tetrad connection coefficients can be expressed more compactly as a two-index quantity defined by

$$N_{ba} \equiv \frac{1}{2}\varepsilon_{acd}\Gamma_{cdb}. \qquad (5)$$

The diagonal components, N_{11}, N_{22}, and N_{33}, describe the twists of the spatial triads along the 1, 2, or 3 directions, respectively. The combinations $N_{ab} + N_{ba}$ of the nondiagonal components represent gravitational wave degrees of freedom. It is sometimes convenient to represent the antisymmetric part of N_{ba} by its own symbol

$$n_a \equiv \frac{1}{2}\varepsilon_{abc}N_{bc}. \qquad (6)$$

The acceleration of the congruence is

$$a_a \equiv \Gamma_{a00}, \qquad (7)$$

and the angular velocity of the spacelike triads relative to Fermi propagated axes is

$$\omega_a \equiv \frac{1}{2}\varepsilon_{abc}\Gamma_{cb0}. \qquad (8)$$

There are nine K_{ba} and nine N_{ba}, giving eighteen primary variables to be evolved. The three a_a and three ω_a are gauge quantities, which, in this paper, are evolved by either the Nester or Lorentz gauge, or kept fixed.

Numerical calculations are performed with a particular choice of coordinates x^μ. Hyperbolic evolution consists in calculating variables on the spacelike hypersurfaces characterized by $x^0 = t_2$ from an initial state specified by values of the variables on an earlier spacelike hypersurface $x^0 = t_1$. Let λ^μ_α denote the transformation matrix between coordinate basis vectors and the tetrad basis vectors:

$$\boldsymbol{e}_\alpha = \lambda^\mu_\alpha \boldsymbol{e}_\mu. \qquad (9)$$

From Eq. (9), we obtain directional derivatives along the tetrad directions in terms of the partial derivatives along the coordinate directions,

$$D_\alpha = \lambda^\mu_\alpha \frac{\partial}{\partial x^\mu}. \qquad (10)$$

The λ^μ_α are the coordinate components of the tetrad basis vectors. The coordinate metric is constructed from these tetrad vector components as $g^{\mu\nu} = \eta^{\alpha\beta}\lambda^\mu_\alpha\lambda^\nu_\beta$.
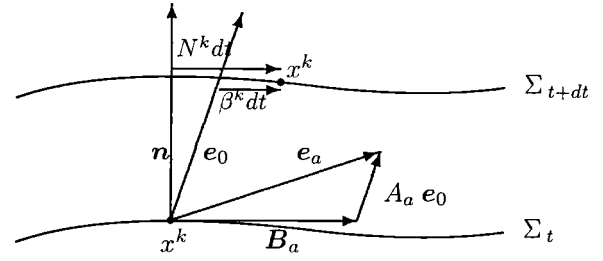

FIG. 1. Decomposition of $\boldsymbol{e}_a$ into a vector tangent to the hypersurface, $\boldsymbol{B}_a$, and a vector parallel to the congruence, $A_a\boldsymbol{e}_0$. Displacements of the spatial coordinates, x^k, relative to the congruence world line equal $\beta^k dt$, where β^k is our tetrad shift vector. Displacements of x^k relative to the hypersurface normal $\boldsymbol{n}$ are $N^k dt$, where N^k is the 3+1 shift vector.

We find it simpler to use as variables not the sixteen λ^μ_α, but the following sixteen quantities which completely determine the tetrad vector components: nine B^k_a, the coordinate components of projections of the spatial tetrad vectors into the hypersurface; three A_a, which measure the tilts of the spatial tetrad vectors relative to the hypersurface and are (minus) the tetrad components of the 3-velocity of the hypersurface frame relative to the tetrad frame; the tetrad lapse function α and the three coordinate components of the tetrad shift vector β^k, which describe the evolution of the coordinates relative to the tetrad congruence. In this paper we evolve the B^k_a and the A_a as dynamic variables, but take α and the β^k to be fixed functions of the coordinates. Eventually we may want to expand the hyperbolic system to include dynamic equations for the lapse and the shift.

In a 3+1 formalism, the lapse function N is the rate of change of proper time with respect to coordinate time along the hypersurface normal, and the shift vector N^k is the rate of displacement of the spatial coordinates with respect to coordinate time relative to the hypersurface normal, such that the coordinate velocity of the normal world line $dx^k/dt = -N^k$. Our tetrad lapse function α is the rate of change of proper time with respect to coordinate time along the tetrad congruence, and our tetrad shift vector β^k is the rate of displacement of the spatial coordinates relative to the tetrad congruence world lines per unit coordinate time. For simple comoving coordinates, the spatial coordinates are constant along the congruence, and $\beta^k = 0$. However, comoving coordinates are not desirable in black hole calculations, since with a finite acceleration tetrad world lines will be continuously advected inward across the event horizon. The tetrad lapse must be chosen so that the constant-time hypersurfaces remain spacelike, which is equivalent to the condition $A_a A_a < 1$.

The projection of $\boldsymbol{e}_a$ into the hypersurface is done along the congruence world lines, such that

$$\boldsymbol{e}_a = A_a\boldsymbol{e}_0 + \boldsymbol{B}_a. \qquad (11)$$

Figure 1 shows how these various vectors are related from the point of view of a frame at rest with respect to the hypersurface.

The directional derivative along a spatial tetrad direction is

$$D_a = A_a D_0 + B_a^k \frac{\partial}{\partial x^k}, \qquad (12)$$

and the directional derivative along the congruence is

$$D_0 = \frac{1}{\alpha} \left(\frac{\partial}{\partial t} - \beta^k \frac{\partial}{\partial x^k} \right). \qquad (13)$$

Plugging Eq. (13) into Eq. (12), we get

$$D_a = \left(B_a^k - \frac{A_a}{\alpha} \beta^k \right) \frac{\partial}{\partial x^k} + \frac{A_a}{\alpha} \frac{\partial}{\partial t}. \qquad (14)$$

Comparing Eqs. (13) and (14) with Eq. (10), we can read off the tetrad vector components. They are

$$\lambda_\alpha^\mu = \begin{bmatrix} \lambda_0^0 & \lambda_0^k \\ \\ \lambda_a^0 & \lambda_a^k \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\alpha} & -\dfrac{\beta^k}{\alpha} \\ \\ \dfrac{A_a}{\alpha} & \left(B_a^k - \dfrac{A_a}{\alpha} \beta^k \right) \end{bmatrix}. \qquad (15)$$

The 3+1 lapse function, N, and shift vector components, N^k, can be expressed in terms of α, β^k, A_a, and B_a^k by constructing g^{tt} and g^{tk} from the tetrad vector components using $g^{\mu\nu} = \eta^{\alpha\beta} \lambda_\alpha^\mu \lambda_\beta^\nu$. With the relations

$$g^{tt} = -\frac{1}{N^2} \quad \text{and} \quad g^{tk} = -\frac{N^k}{N^2}, \qquad (16)$$

we get

$$N = \frac{\alpha}{\sqrt{1 - A_a A_a}}, \qquad (17)$$

$$N^k = \beta^k + \frac{\alpha \ A_a B_a^k}{1 - A_a A_a}. \qquad (18)$$

The tetrad lapse is smaller than the 3+1 lapse due to the time dilation of the tetrad observer in the rest frame of the hypersurface.

III. THE TETRAD EQUATIONS

In this section, we present the evolution and constraint equations for the connection coefficients defined in Sec. II in terms of directional derivative operators along the tetrad directions. The structure of the equations in this form is deceptively simple. One has to keep in mind that partial differential equations are solved in numerical relativity, so the directional derivatives have to be expanded into partial derivatives using Eqs. (13) and (14). In terms of directional derivatives, the equations are quasi-FOSH. We will present

the necessary steps to obtain the true hyperbolic form in Sec. IV. Furthermore, we call the "constraint" equations presented in this section quasiconstraint equations, as is done in [8], because although they contain only spatial directional derivatives, these spatial directionals contain time partials [see Eq. (14)]. The details of converting the quasiconstraint equations into true constraint equations are given in Appendix A.

In both the tetrad and 3+1 approaches, there is an ordering ambiguity inherent in the derivation of a first order hyperbolic evolution system from the Einstein equations. In first order 3+1 formalisms, the spatial derivatives of the metric are independent variables, D_{jkl} (which equal $\frac{1}{2} \partial_j h_{kl}$, where h_{kl} is the spatial metric). Index reordering of the Riemann tensor is an interchange of spatial partial derivatives such that

$$\partial_i D_{jkl} = \partial_j D_{ikl}. \qquad (19)$$

Exploiting this freedom leads to a wide variety of hyperbolic formulations [27]. The standard energy and momentum constraint equations can also be added to the evolution system to get additional hyperbolic or nonhyperbolic structures. In the context of a tetrad approach, the connection coefficients are given in terms of the commutation relations, Eq. (2). To derive integrability conditions from Eq. (2) is messy. We find it is much easier to approach the question of ordering ambiguity from the symmetries of the Riemann tensor, especially since these symmetries are explicit when the indices are all up or down, and it is trivial to raise and lower indices with the Minkowski metric. We do not consider a whole range of schemes as do Kidder, Scheel, and Teukolsky [27], but rather a particular scheme which leads to a simpler hyperbolic structure than the others. To focus on the quasi-FOSH structure of the equations, we present only the principal terms here. The second order source terms are given in Appendix A.

The Riemann tensor projected onto a tetrad is

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= e_\alpha \cdot (\nabla_\gamma \nabla_\delta - \nabla_\delta \nabla_\gamma - \nabla_{[e_\gamma, e_\delta]}) e_\beta \\ &= D_\gamma \Gamma_{\alpha\beta\delta} - D_\delta \Gamma_{\alpha\beta\gamma} + \Gamma_{\alpha\epsilon\gamma} \Gamma^\epsilon_{\beta\delta} - \Gamma_{\alpha\epsilon\delta} \Gamma^\epsilon_{\beta\gamma} \\ &\quad + \Gamma_{\alpha\beta\epsilon} (\Gamma^\epsilon_{\gamma\delta} - \Gamma^\epsilon_{\delta\gamma}), \qquad (20) \end{aligned}$$

where D_α represents directional derivatives along the tetrad directions. The antisymmetry of $R_{\alpha\beta\gamma\delta}$ on the second pair of indices is explicit in Eq. (20). The antisymmetry on the first pair is also trivial because of the antisymmetry of $\Gamma_{\alpha\beta\gamma}$ on the first two indices. However, the Riemann identities

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}, \qquad (21)$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\delta\beta\gamma} + R_{\alpha\gamma\delta\beta} = 0 \qquad (22)$$

lead to new Riemann constraints, which we exploit in the following.

First we derive all the possible quasiconstraint equations, noting the use of Eq. 14.7 from Misner, Thorne, and Wheeler. The energy quasiconstraint equation is

$$G_{00} = R_{1212} + R_{2323} + R_{3131} \sim 2D_a n_a. \qquad (23)$$

An analogous quasiconstraint equation for Ω_a is derived using the cyclic identity, Eq. (22), on $0abc$:

$$0 = R_{0213} + R_{0321} + R_{0132} \sim 2 \ D_a \Omega_a. \qquad (24)$$

The momentum quasiconstraint equations are obtained from $G_{0a} = 0$:

$$G_{01} = R_{0212} + R_{0313}$$
$$\sim -D_1(K_{22} + K_{33}) + D_2 K_{12} + D_3 K_{13}, \qquad (25)$$

$$G_{02} = R_{0121} + R_{0323}$$
$$\sim -D_2(K_{11} + K_{33}) + D_1 K_{21} + D_3 K_{23}, \qquad (26)$$

$$G_{03} = R_{0131} + R_{0232}$$
$$\sim -D_3(K_{11} + K_{22}) + D_1 K_{31} + D_2 K_{32}. \qquad (27)$$

Similar quasiconstraint equations involving N_{ab} are derived solely from Eq. (21) applied to spatial indices of the Riemann tensor:

$$R_{1213} - R_{1312} \sim D_1(N_{22} + N_{33}) - D_2 N_{12} - D_3 N_{13}, \quad (28)$$

$$R_{2321} - R_{2123} \sim D_2(N_{11} + N_{33}) - D_1 N_{21} - D_3 N_{23}, \quad (29)$$

$$R_{3132} - R_{3231} \sim D_3(N_{11} + N_{22}) - D_1 N_{31} - D_2 N_{32}. \quad (30)$$

We now to turn to evolution equations. By using Eq. (21) in the momentum quasi-constraint equations, we calculate *evolution* equations for the nondiagonal components of N_{ab}. For example, interchanging the first and second pairs of indices in Eq. (27) as shown below gives evolution equations for N_{21} and N_{12}:

$$G_{03} = R_{0131} + R_{3202}$$
$$\sim -D_0 N_{21} - D_2 \omega_1 - D_3 K_{11} + D_1 K_{31}, \qquad (31)$$

$$G_{03} = R_{3101} + R_{0232}$$
$$\sim D_0 N_{12} + D_1 \omega_2 + D_2 K_{32} - D_3 K_{22}. \qquad (32)$$

Evolution equations for the diagonal components of N_{ab} are obtained solely from applications of Eq. (21):

$$R_{2301} - R_{0123} \sim D_0 N_{11} + D_1 \omega_1 + D_2 K_{31} - D_3 K_{21}, \quad (33)$$

$$R_{3102} - R_{0231} \sim D_0 N_{22} + D_2 \omega_2 + D_3 K_{12} - D_1 K_{32}, \quad (34)$$

$$R_{1203} - R_{0312} \sim D_0 N_{33} + D_3 \omega_3 + D_1 K_{23} - D_2 K_{13}. \quad (35)$$

Evolution equations for K_{ab} are obtained from the Einstein equations. As is done elsewhere (see for example [28]), a multiple of the energy quasiconstraint equation times the spatial metric can be subtracted from the K_{ab} evolution equations. In the formulation we are presenting, this amounts to subtracting a multiple of Eq. (23) from the diagonal spatial components of the Ricci tensor, since our metric is just δ_a^b.

The same type of procedure in 3+1 formulations affects the evolution equations for the nondiagonal as well as the diagonal components of K_{ab}, since the spatial metric there is, in general, nondiagonal. The number we choose to multiply the energy quasiconstraint equation by is 1, to bring the evolution equations for the diagonal K_{ab} into the same form as those for the nondiagonal components. With the aid of Eq. (21), we derive sample evolution equations for K_{ab} below:

$$R_{12} = -R_{1020} + R_{1323} = -R_{1020} + R_{2313}$$
$$\sim D_0 K_{21} - D_2 a_1 - D_3 N_{11} + D_1 N_{31}, \qquad (36)$$

$$R_{12} = -R_{2010} + R_{1323}$$
$$\sim D_0 K_{12} - D_1 a_2 - D_2 N_{32} + D_3 N_{22}, \qquad (37)$$

$$R_{11} - G_{00} = -R_{0101} - R_{2323}$$
$$\sim D_0 K_{11} - D_1 a_1 - D_2 N_{31} + D_3 N_{21}. \qquad (38)$$

If the gauge quantities a_a and ω_a are fixed functions of time, then the quasi-FOSH structure is now complete and is represented by the following system of eighteen equations for the nine K_{ab} and nine N_{ab}:

$$D_0 K_{ab} - D_a a_b - \varepsilon_{acd} D_c N_{db} = S\_K_{ab}, \qquad (39)$$

$$D_0 N_{ab} + D_a \omega_b + \varepsilon_{acd} D_c K_{db} = S\_N_{ab}, \qquad (40)$$

where S_K_{ab} and S_N_{ab} are source terms quadratic in the connection coefficient variables. (See Appendix B.)

Alternatively, one can implement a dynamic gauge. Both the Nester gauge used by ERW and EW and the Lorentz gauge used by van Putten and Eardley result in evolution equations for a_a and ω_a which, when added to Eqs. (39) and (40), form a quasi-FOSH system.

The Nester gauge conditions in [9] are defined for an arbitrary number of dimensions. In 4D spacetime, the Nester conditions state that two 1-forms, $\tilde{q}$ and $\hat{q}$, whose tetrad components are

$$\tilde{q}_\alpha = \Gamma_\alpha{}^\gamma{}_\gamma, \quad \hat{q}_\alpha = \varepsilon_{\alpha\beta\gamma\delta} \Gamma^{\beta\gamma\delta}, \qquad (41)$$

are closed. This implies vanishing exterior derivatives so that, in tetrad component form, the Nester conditions are

$$\nabla_\alpha \tilde{q}_\beta - \nabla_\beta \tilde{q}_\alpha = 0, \qquad (42)$$

$$\nabla_\alpha \hat{q}_\beta - \nabla_\beta \hat{q}_\alpha = 0. \qquad (43)$$

The Nester conditions result in six evolution equations,

$$D_0 a_b - D_c K_{cb} = -(\omega_c - 2\Omega_c)N_{cb} + (\mathrm{Tr}\,N)\omega_b,$$

$$D_0 \omega_b + D_c N_{cb} = -(\omega_c - 2\Omega_c)K_{cb} - (\mathrm{Tr}\,N)a_b, \quad (44)$$

and six quasiconstraint equations,

$$\varepsilon_{abc} D_b(a_c - 2 \ n_c) = 2(\mathrm{Tr}\,K)\Omega_a - (\mathrm{Tr}\,N)(a_a - 2n_a)$$
$$+ (a_d - 2 \ n_d)N_{da},$$

$$\varepsilon_{abc}D_b(\omega_c - 2\ \Omega_c) = -(\operatorname{Tr}N)\omega_a + (\omega_d - 2\Omega_d)N_{da}.$$
(45)

The Lorentz gauge condition in [7] is $\nabla^\gamma\omega_{\gamma\alpha\beta}=0$, where $\omega_{\gamma\alpha\beta}=\Gamma_{\alpha\beta\gamma}$. For fixed α and β, the $\omega_{\gamma\alpha\beta}$ are the components of connection 1-forms (see Wald [24]). Expressed in terms of $\Gamma_{\alpha\beta\gamma}$, the Lorentz gauge condition is

$$D_\delta\Gamma_{\alpha\beta}{}^\delta + \Gamma_{\alpha\beta}{}^\gamma\Gamma^\delta{}_{\gamma\delta} = 0.$$
(46)

The Lorentz gauge results in six evolution equations:

$$D_0 a_b - D_c K_{cb} = (a_c - 2\ n_c)K_{cb} - (\operatorname{Tr}K)a_b,$$

$$D_0\omega_b + D_c N_{cb} = -(a_c - 2\ n_c)N_{cb} - (\operatorname{Tr}K)\omega_b,$$
(47)

and no additional constraint equations. Note that the principal terms in Eq. (47) are identical to those for the Nester gauge evolution equations, Eq. (44).

The evolution equations expressed in terms of directional derivatives, Eqs. (39), (40), and (with a dynamic gauge) (44) or (47), can be written in a condensed notation,

$$D_0\mathbf{q} + \mathbf{M}^a D_a\mathbf{q} = \mathbf{S},$$
(48)

where $\mathbf{q}$ is a vector of the twenty-four variables N_{ab}, K_{ab}, a_a, and ω_a. Furthermore, $\mathbf{M}^a$ are three sparse 24×24 matrices whose only nonzero elements are ±1, and $\mathbf{S}$ is a vector of source terms. If one orders the variables so that

$$\mathbf{q} = (N_{11}, N_{21}, N_{31}, a_1, K_{11}, K_{21}, K_{31}, \omega_1, N_{12}, N_{22}, N_{32}, a_2,$$
$$K_{12}, K_{22}, K_{32}, \omega_2, N_{13}, N_{23}, N_{33}, a_3, K_{13}, K_{23}, K_{33}, \omega_3),$$
(49)

then the $\mathbf{M}^a$ matrices have a simple block diagonal structure. Each is composed of three identical 8×8 blocks, respectively:

$$\mathbf{M}^1_{block} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
(50)

$$\mathbf{M}^2_{block} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
(51)

$$\mathbf{M}^3_{block} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$
(52)

IV. HYPERBOLIC STRUCTURE OF THE COORDINATE EQUATIONS

The beautiful quasi-FOSH tetrad formulation with constant coefficients given in Eqs. (48) to (52) is not so simple, or so beautiful, when expressed in terms of coordinates. However, we proceed to show that the coordinate equations are still quite manageable and are, indeed, symmetrizable hyperbolic.

Substituting Eqs. (13) and (14) into Eq. (48), we express the tetrad evolution equations in terms of partial derivatives along coordinate directions:

$$\mathbf{T}D_0\mathbf{q} + \mathbf{M}^a B_a^k \frac{\partial\mathbf{q}}{\partial x^k} = \mathbf{S},$$
(53)

where

$$\mathbf{T} \equiv [\mathbf{I} + \mathbf{M}^a A_a],$$
(54)

and $\mathbf{I}$ is the identity matrix. For clarity of notation, we let D_0 represent the partials in Eq. (13). The $\mathbf{T}$ matrix has a block diagonal structure composed of three identical 8×8 entries which are

$$
\boldsymbol{T}_{block} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -A_3 & A_2 & A_1 \\
0 & 1 & 0 & 0 & A_3 & 0 & -A_1 & A_2 \\
0 & 0 & 1 & 0 & -A_2 & A_1 & 0 & A_3 \\
0 & 0 & 0 & 1 & -A_1 & -A_2 & -A_3 & 0 \\
0 & A_3 & -A_2 & -A_1 & 1 & 0 & 0 & 0 \\
-A_3 & 0 & A_1 & -A_2 & 0 & 1 & 0 & 0 \\
A_2 & -A_1 & 0 & -A_3 & 0 & 0 & 1 & 0 \\
A_1 & A_2 & A_3 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\tag{55}
$$

We now multiply Eq. (53) by $\boldsymbol{T}^{-1}$ to give

$$
D_0 \mathbf{q} + \boldsymbol{C}^a B_a^k \frac{\partial \mathbf{q}}{\partial x^k} = \boldsymbol{T}^{-1} \mathbf{S},
\tag{56}
$$

where

$$
\boldsymbol{C}^a \equiv \boldsymbol{T}^{-1} \boldsymbol{M}^a.
\tag{57}
$$

$\boldsymbol{T}^{-1}$ is straightforward to calculate, since it is a block diagonal matrix with each block equal to the inverse of $\boldsymbol{T}_{block}$ in Eq. (55), where

$$
\boldsymbol{T}_{block}^{-1} = \frac{1}{1 - A_a A_a}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & A_3 & -A_2 & -A_1 \\
0 & 1 & 0 & 0 & -A_3 & 0 & A_1 & -A_2 \\
0 & 0 & 1 & 0 & A_2 & -A_1 & 0 & -A_3 \\
0 & 0 & 0 & 1 & A_1 & A_2 & A_3 & 0 \\
0 & -A_3 & A_2 & A_1 & 1 & 0 & 0 & 0 \\
A_3 & 0 & -A_1 & A_2 & 0 & 1 & 0 & 0 \\
-A_2 & A_1 & 0 & A_3 & 0 & 0 & 1 & 0 \\
-A_1 & -A_2 & -A_3 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\tag{58}
$$

The system given in Eq. (56) is hyperbolic according to the definition given in [11] if any linear combination $b_a \boldsymbol{C}^a$ of $\boldsymbol{C}^1$, $\boldsymbol{C}^2$, and $\boldsymbol{C}^3$ can be diagonalized with a complete set of eigenvectors and real eigenvalues. The linear combination for propagation in the x^k coordinate direction has $b_a = B_a^k$. Solving the eigensystem of the combined 8×8 matrix in MATHEMATICA gives for the eigenvalues

$$
\frac{\boldsymbol{b} \cdot \boldsymbol{A} \mp \sqrt{\boldsymbol{b} \cdot \boldsymbol{b} - (\boldsymbol{b} \times \boldsymbol{A}) \cdot (\boldsymbol{b} \times \boldsymbol{A})}}{1 - \boldsymbol{A} \cdot \boldsymbol{A}},
\tag{59}
$$

where $\boldsymbol{b}$ and $\boldsymbol{A}$ are the 3-vectors in the spatial orthonormal frame with components b_a and A_a, respectively. The dot and cross products are the standard vector operations, and the upper or lower signs on the square root are for left or right-propagating (relative to $\boldsymbol{b}$) modes. The eigenvalues are real as long as $\boldsymbol{A} \cdot \boldsymbol{A} = A_a A_a < 1$. The eigenvectors (given in Appendix B) form a complete set. The lapse and shift hidden in the D_0 operator modify the eigenvalues in a trivial way (multiply by the lapse α, then subtract the shift β^k), but have no effect on the eigenvectors.

For Eq. (56) to also be symmetrizable by the definition of Lindblom and Scheel [12], a positive definite symmetric matrix must be found which multiplies the $\boldsymbol{C}^a$ matrices to give symmetric matrices. The obvious candidate for such a symmetrizer is $\boldsymbol{T}$, since $\boldsymbol{T} \boldsymbol{C}^a = \boldsymbol{M}^a$ by Eq. (57), and the $\boldsymbol{M}^a$ matrices are symmetric. $\boldsymbol{T}$ is real and symmetric so a necessary and sufficient condition for it to be positive definite is that all its eigenvalues are positive [29]. This requires that $A_a A_a < 1$.

Note that our saying the system is hyperbolic is contingent on the evolution of the B_a^k being hyperbolic. This is discussed further in Sec. V.

V. EVOLUTION AND CONSTRAINT EQUATIONS FOR A_a AND B_a^k

The commutators of the orthonormal basis vectors are

$$
[\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta] = \nabla_\alpha \boldsymbol{e}_\beta - \nabla_\beta \boldsymbol{e}_\alpha = (\Gamma_{\beta\alpha}^\gamma - \Gamma_{\alpha\beta}^\gamma) \boldsymbol{e}_\gamma.
\tag{60}
$$

Expressed in terms of tetrad components and partial derivatives,

$$[e_\alpha, e_\beta] = \left(\lambda_\alpha^\mu \frac{\partial \lambda_\beta^\nu}{\partial x^\mu} - \lambda_\beta^\mu \frac{\partial \lambda_\alpha^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu}. \tag{61}$$

Expanding Eq. (61) and collecting terms gives

$$\left(\lambda_\alpha^t \frac{\partial \lambda_\beta^k}{\partial t} + \lambda_\alpha^k \frac{\partial \lambda_\beta^k}{\partial x^k} - \lambda_\beta^t \frac{\partial \lambda_\alpha^k}{\partial t} - \lambda_\beta^k \frac{\partial \lambda_\alpha^k}{\partial x^k} \right) \frac{\partial}{\partial x^k}$$

$$+ \left(\lambda_\alpha^t \frac{\partial \lambda_\beta^t}{\partial t} + \lambda_\alpha^k \frac{\partial \lambda_\beta^t}{\partial x^k} - \lambda_\beta^t \frac{\partial \lambda_\alpha^t}{\partial t} - \lambda_\beta^k \frac{\partial \lambda_\alpha^t}{\partial x^k} \right) \frac{\partial}{\partial t}. \tag{62}$$

Doing the same in Eq. (60), we get

$$[(\Gamma^0{}_{\beta\alpha} - \Gamma^0{}_{\alpha\beta})\lambda_0^k + (\Gamma^c{}_{\beta\alpha} - \Gamma^c{}_{\alpha\beta})\lambda_c^k] \frac{\partial}{\partial x^k}$$

$$+ [(\Gamma^0{}_{\beta\alpha} - \Gamma^0{}_{\alpha\beta})\lambda_0^t + (\Gamma^c{}_{\beta\alpha} - \Gamma^c{}_{\alpha\beta})\lambda_c^t] \frac{\partial}{\partial t}. \tag{63}$$

Set Eqs. (62) and (63) equal, and let the index $\alpha = 0$ and the index $\beta = a$. Simplifying, we obtain evolution equations for B_a^k and A_a:

$$D_0 B_a^k + \frac{B_a^l}{\alpha} \frac{\partial \beta^k}{\partial x^l} = \frac{1}{\alpha} \left(\frac{\partial}{\partial t} - \mathcal{L}_\beta \right) B_a^k$$

$$= -\varepsilon_{abc} \omega_b B_c^k - K_{ac} B_c^k, \tag{64}$$

$$D_0 A_a = a_a - \varepsilon_{abc} \omega_b A_c - K_{ac} A_c - \frac{B_a^l}{\alpha} \frac{\partial \alpha}{\partial x^l}, \tag{65}$$

where $\mathcal{L}_\beta$ in Eq. (64) is the Lie derivative. For fixed lapse and shift, the evolution of both the B_a^k and the A_a is just advection along the congruence world lines and trivially hyperbolic.

The congruence can always be chosen to be orthogonal to the initial hypersurface, so $A_a = 0$ initially. However, A_a will not remain zero during the subsequent evolution unless the condition

$$B_a^l \frac{\partial}{\partial x^l} \log \alpha = a_a \tag{66}$$

is satisfied at all times. For either the fixed or the dynamic gauge conditions on a_a considered here, the evolution of B_a^l and/or a_a is inconsistent with Eq. (66), except possibly for very special initial conditions.

Repeating the same process as above, but with the index $\alpha = a$ and the index $\beta = b$, we obtain constraint equations for B_a^k and A_a:

$$\varepsilon_{cab} B_a^l \frac{\partial B_b^k}{\partial x^l} = N_{dc} B_d^k - (\operatorname{Tr} N) B_c^k + \varepsilon_{cab} A_a (\varepsilon_{bdf} \omega_d B_f^k$$

$$+ K_{bd} B_d^k), \tag{67}$$

$$\varepsilon_{cab} B_a^l \frac{\partial A_b}{\partial x^l} = 2\Omega_c + N_{dc} A_d - (\operatorname{Tr} N) A_c + \varepsilon_{cab} A_a$$

$$\times (-a_b + \varepsilon_{bdf} \omega_d A_f + K_{bd} A_d). \tag{68}$$

Note that if $A_a = 0$, the constraint of Eq. (68) is satisfied automatically, and Eq. (67) can be used to calculate all of the N_{ab} from the B_a^k.

VI. THE INITIAL VALUE PROBLEM

Initial conditions for the variables must be chosen so all relevant constraints are satisfied on the initial hypersurface. This is most easily accomplished if the initial tetrad is oriented so the tetrad congruence is orthogonal to the initial hypersurface. Then, $\Omega_a = 0$ and $A_a = 0$ initially. The initial B_a^k are the components of an orthonormal triad of vectors lying in the initial hypersurface, related to the inverse of the spatial metric of the hypersurface by $h^{kl} = B_a^k B_a^l$. One way to construct consistent initial conditions is to solve the initial value problem using standard 3+1 methods for the spatial metric and the extrinsic curvature. Construct orthonormal triad fields by a Gramm-Schmidt orthogonalization procedure, orienting the B_1 vector along the x^1 coordinate direction, and B_2 in the $x^1 - x^2$ plane, for instance. Find the N_{ab} from the commutators of the B_a 3-vectors using Eq. (67). The K_{ab} are simply the projections of the coordinate components of the extrinsic curvature as found from the 3+1 initial value problem along the orthonormal triad vectors. This procedure is guaranteed to give consistent initial conditions for the tetrad vectors, as long as the spatial coordinates are basically Cartesian, i.e., there are no spatial coordinate singularities. The initial acceleration and angular velocity of the tetrad are arbitrary, except in the context of the Nester gauge. In this case, the initial angular velocity ω_a and the initial $a_a - 2 n_a$ must have vanishing exterior derivatives. However, there is no guarantee that there are not large twists of the initial triad vectors, possibly leading to large gauge transients in the context of one of the dynamical gauge conditions.

A more elegant and, likely, a better-behaved choice for the initial spatial triad is to require that it satisfy the 3D Nester gauge conditions in the hypersurface. These conditions are that the 3D one-form $\tilde{q}_a$, whose triad components are $\Gamma_a{}^b{}_b$, has zero exterior derivative, and that the trace of the N_{ab} matrix vanish, i.e. $N_{11} + N_{22} + N_{33} = 0$. For simple topologies, the first condition is equivalent to the condition that $2n_a = \varepsilon_{abc} N_{bc}$ be the gradient of a scalar. Finding a solution for the B_a^k which satisfies these conditions is, in general, a nontrivial elliptic problem (see [30] and [31]). However, the situation is much simpler for conformally flat 3-geometries. Taking Cartesian basis vectors in the conformal geometry, the conformal N_{ab} are zero. If the conformal factor which

generates the physical metric is $e^{4\psi}$, the conformal transformation simply rescales the B_a^k by a factor $e^{-2\psi}$. The physical n_a equal $-2D_a\psi$ (the gradient of a scalar), and the symmetric part of N_{ab} is still zero. The 3D Nester condition is satisfied.

VII. DISCUSSION

Although the emphasis of the tetrad formalisms in the literature is on evolving spacetimes with physically defined flows, we think a tetrad formalism may be useful for vacuum numerical calculations of black hole spacetimes. The tetrad formulation we have presented, based on that of Estabrook, Robinson, and Wahlquist [3], allows control of the evolution of the timelike congruence through either dynamic or fixed gauge conditions on a_a and ω_a. The lapse and shift we have introduced allow for a completely arbitrary evolution of the coordinates with minimal complication of the formalism. Furthermore, since the variables evolved are defined relative to the orthonormal frames, the metric is the Minkowski metric and there are no nonlinearities in the equations associated with the inverse metric. The system of equations based on coordinate derivatives is symmetrizable hyperbolic, though admittedly more complicated than the quasi-FOSH system based on directional derivatives. Finally, the variables N_{ab}, K_{ab}, a_a and ω_a are all scalars, so derivatives of the shift only appear in the evolution equations for B_a^k. We have successfully implemented this tetrad formalism for 1D colliding gravitational plane waves [32], where it results in substantially better accuracy and stability compared with our calculations in the 3+1 framework [2].

There are still important issues to be worked out in order to apply this tetrad formalism to 3D black hole codes. First, one must deal with the complications that arise when evolving the congruence as well as the hypersurface. The congruence stays timelike as long as the acceleration is bounded; however, in order for the hypersurfaces to remain spacelike, the condition $A_a A_a < 1$ must be satisfied at all times. If $A_a A_a = 1$, the system breaks down completely. The hypersurfaces become null, causing the 3+1 lapse to blow up. In addition, the coordinate equations become singular (see Appendix B). $A_a A_a$ can be kept small by an adjustment of the lapse as the calculation proceeds. Such a resetting of the lapse invalidates theorems which bound the growth of the solution based on the symmetrizable hyperbolic structure of the system, but does not affect other advantages of the hyperbolic formulation, such as dealing with boundary conditions.

Second, attaining a stationary solution at late times for general 3D black hole numerical calculations may be problematic in a tetrad formulation. Such solutions are potentially desirable for long-term stability. With bounded acceleration, congruence world lines just outside the apparent horizon will necessarily cross the horizon and be trapped by the black hole. World lines further out may actually accelerate outward relative to a static or stationary observer. While a shift adjusted just right may keep grid stretching under control in spite of this, the adjustment will have to be time-dependent except for very special initial conditions on the congruence.

Knowing what these special initial conditions are requires knowing the whole future solution ahead of time, and thus is not practical.

Both of these issues will probably best be addressed by extending the symmetrizable hyperbolic system to allow for a dynamic lapse, to keep $A_a A_a$ small, and a dynamic shift, to control grid stretching.

APPENDIX A: SOURCE TERMS OF TETRAD EVOLUTION AND QUASI-CONSTRAINT EQUATIONS

The tetrad energy quasiconstraint equation, derived from $G_{00}=0$, is

$$2D_a n_a = -2\omega_a \Omega_a + N_{cd}N_{cd} + \frac{1}{2}(K_{cd}K_{dc} - (\mathrm{Tr}\,K)^2$$
$$- N_{cd}N_{dc} - (\mathrm{Tr}\,N)^2). \tag{A1}$$

The tetrad momentum quasiconstraint equations, obtained from $G_{0a}=0$, are

$$D_1(K_{22}+K_{33}) - D_2 K_{12} - D_3 K_{13}$$
$$= 2\varepsilon_{1bc}a_b\ \Omega_c + \varepsilon_{1bc}K_{bd}N_{dc} - 2K_{1c}n_c, \tag{A2}$$

$$D_2(K_{11}+K_{33}) - D_1 K_{21} - D_3 K_{23}$$
$$= 2\varepsilon_{2bc}a_b\Omega_c + \varepsilon_{2bc}K_{bd}N_{dc} - 2K_{2c}n_c, \tag{A3}$$

$$D_3(K_{11}+K_{22}) - D_1 K_{31} - D_2 K_{32}$$
$$= 2\varepsilon_{3bc}a_b\Omega_c + \varepsilon_{3bc}K_{bd}N_{dc} - 2K_{3c}n_c. \tag{A4}$$

Analogous quasiconstraint equations for Ω_a and N_{ab} are

$$2D_a\Omega_a = 2a_a\Omega_a + 4n_a\Omega_a, \tag{A5}$$

$$D_1(N_{22}+N_{33}) - D_2 N_{12} - D_3 N_{13}$$
$$= -2\varepsilon_{1bc}\omega_b\Omega_c + 2K_{1c}\Omega_c - \varepsilon_{1bc}N_{1b}N_{c1}$$
$$+ 2n_1(N_{22}+N_{33}), \tag{A6}$$

$$D_2(N_{11}+N_{33}) - D_1 N_{21} - D_3 N_{23}$$
$$= -2\varepsilon_{2bc}\omega_b\Omega_c + 2K_{2c}\Omega_c - \varepsilon_{2bc}N_{2b}N_{c2}$$
$$+ 2n_2(N_{11}+N_{33}), \tag{A7}$$

$$D_3(N_{11}+N_{22}) - D_1 N_{31} - D_2 N_{32}$$
$$= -2\varepsilon_{3bc}\omega_b\Omega_c + 2K_{3c}\Omega_c$$
$$- \varepsilon_{3bc}N_{3b}N_{c3} + 2n_3(N_{11}+N_{22}). \tag{A8}$$

To obtain the true constraint equations from these quasi-constraint equations, Eq. (12) must be substituted for each appearance of D_a. Then Eq. (56) must be used to eliminate the terms involving D_0 that have been introduced. The true constraints are only needed to check accuracy as the solution evolves, once consistent initial data is obtained.

The sources for K_{ab} and N_{ab} evolution Eqs. (39) and (40) are

$$S\_K_{ab} = a_a a_b + \varepsilon_{bcd}(-N_{ac}a_d + K_{ac}\omega_d) + \varepsilon_{acd}K_{cb}\omega_d$$

$$+ \frac{1}{2}\varepsilon_{adf}\varepsilon_{bce}(K_{dc}K_{fe} - N_{dc}N_{fe}) + (\mathrm{Tr}\,N)N_{ab}$$

$$- N_{ca}N_{cb} - K_{ac}K_{cb} + 2\omega_b\Omega_a, \qquad (A9)$$

$$S\_N_{ab} = -a_a\omega_b + \varepsilon_{bcd}(K_{ac}a_d + N_{ac}\omega_d) + \varepsilon_{acd}N_{cb}\omega_d$$

$$+ \varepsilon_{adf}\varepsilon_{bce}N_{dc}K_{fe} - (\mathrm{Tr}\,N)K_{ab} + N_{ca}K_{cb} - K_{ac}N_{cb}$$

$$+ 2a_b\Omega_a. \qquad (A10)$$

APPENDIX B: DETAILS OF COORDINATE EQUATIONS

In this appendix, we expand Eq. (56). Because of the block diagonal structure of the C^a and M^a matrices, it is only necessary to work with one set of eight variables. We will let the free index $d = 1$, 2, or 3 (for the first, second, and third set of eight variables in the vector $\mathbf{q}$), i.e.

$$\mathbf{q}_d = (N_{1d}, N_{2d}, N_{3d}, a_d, K_{1d}, K_{2d}, K_{3d}, \omega_d). \qquad (B1)$$

So as to include a nonzero shift with minimal notation, we use D_0 to represent the partial derivatives in Eq. (13). Then we can write Eq. (56) as

$$D_0\mathbf{q}_d + [C^1_{block}B^k_1 + C^2_{block}B^k_2 + C^3_{block}B^k_3]\frac{\partial\mathbf{q}_d}{\partial x^k} = \mathbf{S}_d, \qquad (B2)$$

where

$$C^1_{block} = \frac{1}{1 - A_a A_a}
\begin{bmatrix}
A_1 & -A_2 & -A_3 & 0 & 0 & 0 & 0 & -1 \\
A_2 & A_1 & 0 & -A_3 & 0 & 0 & 1 & 0 \\
A_3 & 0 & A_1 & A_2 & 0 & -1 & 0 & 0 \\
0 & A_3 & -A_2 & A_1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & A_1 & -A_2 & -A_3 & 0 \\
0 & 0 & -1 & 0 & A_2 & A_1 & 0 & -A_3 \\
0 & 1 & 0 & 0 & A_3 & 0 & A_1 & A_2 \\
-1 & 0 & 0 & 0 & 0 & A_3 & -A_2 & A_1
\end{bmatrix}, \qquad (B3)$$

$$C^2_{block} = \frac{1}{1 - A_a A_a}
\begin{bmatrix}
A_2 & A_1 & 0 & A_3 & 0 & 0 & -1 & 0 \\
-A_1 & A_2 & -A_3 & 0 & 0 & 0 & 0 & -1 \\
0 & A_3 & A_2 & -A_1 & 1 & 0 & 0 & 0 \\
-A_3 & 0 & A_1 & A_2 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & A_2 & A_1 & 0 & A_3 \\
0 & 0 & 0 & 1 & -A_1 & A_2 & -A_3 & 0 \\
-1 & 0 & 0 & 0 & 0 & A_3 & A_2 & -A_1 \\
0 & -1 & 0 & 0 & -A_3 & 0 & A_1 & A_2
\end{bmatrix}, \qquad (B4)$$

$$C^3_{block} = \frac{1}{1 - A_a A_a}
\begin{bmatrix}
A_3 & 0 & A_1 & -A_2 & 0 & 1 & 0 & 0 \\
0 & A_3 & A_2 & A_1 & -1 & 0 & 0 & 0 \\
-A_1 & -A_2 & A_3 & 0 & 0 & 0 & 0 & -1 \\
A_2 & -A_1 & 0 & A_3 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & A_3 & 0 & A_1 & -A_2 \\
1 & 0 & 0 & 0 & 0 & A_3 & A_2 & A_1 \\
0 & 0 & 0 & 1 & -A_1 & -A_2 & A_3 & 0 \\
0 & 0 & -1 & 0 & A_2 & -A_1 & 0 & A_3
\end{bmatrix}, \qquad (B5)$$

and

$$\mathbf{S}_d = \frac{1}{1-A_a A_a} \begin{bmatrix} A_3 S\_K_{2d} - A_2 \ S\_K_{3d} + S\_N_{1d} - A_1 \ S\_\omega_d \\ -A_3 S\_K_{1d} + A_1 \ S\_K_{3d} + S\_N_{2d} - A_2 \ S\_\omega_d \\ A_2 S\_K_{1d} - A_1 \ S\_K_{2d} + S\_N_{3d} - A_3 \ S\_\omega_d \\ S\_a_d + A_1 \ S\_K_{1d} + A_2 \ S\_K_{2d} + A_3 \ S\_K_{3d} \\ A_1 S\_a_d + S\_K_{1d} - A_3 \ S\_N_{2d} + A_2 \ S\_N_{3d} \\ A_2 S\_a_d + S\_K_{2d} + A_3 \ S\_N_{1d} - A_1 \ S\_N_{3d} \\ A_3 S\_a_d + S\_K_{3d} - A_2 \ S\_N_{1d} + A_1 \ S\_N_{2d} \\ -A_1 S\_N_{1d} - A_2 \ S\_N_{2d} - A_3 \ S\_N_{3d} + S\_\omega_d \end{bmatrix}. \tag{B6}$$

The expressions for S_N_{1d}, S_N_{2d}, S_N_{3d}, S_K_{1d}, S_K_{2d}, S_K_{3d} in Eq. (B6) are obtained from Eqs. (A9) and (A10). Those for S_a_d and S_ω_d are from either Eqs. (44) or Eqs. (47).

The eight eigenvectors of the arbitrary linear combination $b_a \, \mathbf{C}^a_{block}$ consist of four pairs of left and right-propagating modes. Each pair can be chosen to involve only one of the four variables K_{1d}, K_{2d}, K_{3d}, ω_d. For propagation in the x^k coordinate direction, $b_a = B^k_a$. As in Sec. IV, we simplify notation by using $\boldsymbol{b}$ and $\boldsymbol{A}$ to denote the 3-vectors in the spatial orthonormal frame with components b_a and A_a. Then the ω_d eigenvectors, normalized so $\omega_d = \boldsymbol{b} \cdot \boldsymbol{b} = |\boldsymbol{b}|^2$, have $N_{bd} = [\boldsymbol{b} \times (\boldsymbol{b} \times \boldsymbol{A})]_b \pm b_b \sqrt{|\boldsymbol{b}|^2 - |\boldsymbol{b} \times \boldsymbol{A}|^2}$, and $a_d = 0$. The K_{ad} eigenvectors, normalized so $K_{ad} = |\boldsymbol{b}|^2$, have $N_{bd} = \varepsilon_{abc}([\boldsymbol{b} \times (\boldsymbol{b} \times \boldsymbol{A})]_c \pm b_c \sqrt{|\boldsymbol{b}|^2 - |\boldsymbol{b} \times \boldsymbol{A}|^2})$ and $a_d = -([\boldsymbol{b} \times (\boldsymbol{b} \times \boldsymbol{A})]_a \pm b_a \sqrt{|\boldsymbol{b}|^2 - |\boldsymbol{b} \times \boldsymbol{A}|^2})$. The upper sign on the square root corresponds to the upper sign in Eq. (59) for the eigenvalues. Note that N_{ad} is zero in the K_{ad} eigenvectors.

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